How to pierce tank armor. Lavrentiev's theory of shaped explosive

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Here is another nice example of application of conformal maps taken from [1, 2, 3].

The following phenomenon was discovered by the American physicist Charles Edward Munroe (1880).

You put a cylindrical charge of high explosive (15 cm high, 4 cm diameter) on top of a thick armor plate (20 cm thickness) and explode it. You get a little dent (about 1 cm deep) in the plate. Now explode it at some height (6 cm) over the plate. There will be almost no trace on the plate surface. Now drill a conical cavity, opening downward, in the explosive charge from the bottom side of it. The effect increases: you obtain a little pit few centimeters deep. This is already a surprise: you used *less* explosive than in the first experiment and you achieve a stronger effect. Now cover the surface of the conical cavity by a thin (1.5 mm) layer of steel. You get a pit in the armor about 5 cm deep. And finally explode the charge with the conical cavity covered by steel at the height of 6 cm over the plate. This time you pierce the plate through!

The physical explanation is that the conical cavity in the explosive forms a thin jet made of the steel covering the cavity, and this jet has so high kinetic energy, because of its velocity, that it pierces a thick armor plate.

The effect is widely used since WWII in anti-tank weapons. The tanks are covered by armor plates, several centimeters thick. Normally, to pierce this armor by kinetic energy of a heavy shell you need very high velocity. This is achieved by a heavy cannon, a very bulky and expensive thing.

(Most of the images found in Google under "anti-tank gun" show small caliber guns and even rifles which were obsolete by the early 1940-s, as the tank armor improved. To penetrate the armor of a modern tank (after 1943) one needs a big gun, of 100 mm and higher caliber.)

Using the Munroe effect you can pierce the same plate with a grenade whose velocity is irrelevant: you can throw it by hand, or shoot from a small portable rocket launcher. This sort of weapon was developed before and during WWII. (US Bazooka, Soviet RPG (=Rocket Anti-tank Grenade launcher), Soviet anti-tank hand grenade, and German Panzerfaust. You can see many pictures in Wikipedia and movies on U-Tube).

In Russian literature, this jet of steel formed by explosion, which pierces the armor, is called the "wire". It is very thin indeed. You can see the holes made by these shaped charges in the tanks on display in museums. You can hardly stick a pencil into these holes, but they really penetrate the whole thickness of an armor plate (and ignite everything which can burn inside). So for example the picture http://en.wikipedia.org/wiki/File:Tali-Ihantala.jpg is misleading: the tank in the background was destroyed by another kind of weapon, definitely not by the Panzerfaust grenades that soldiers in the foreground hold.

Actually it is relatively easy to protect a tank from these shaped charge projectiles. One simply adds on the outside of the armor some thick but light layer. This can be a thin sheet metal placed at some distance outside the main armor. It will cause the projectile explode at a larger distance, so that the jet does not reach the armor.

It is much more difficult, practically impossible, to protect against real heavy high velocity cannon shells.

Wikipedia says that traditional large caliber, high velocity anti-tank guns are used nowadays only by the Russians, and "some other countries", which means that all anti-tank weapons produced in the West is based on the cumulative effect.

A mathematical theory of Munroe effect (which is called cumulative effect in Russian) was developed by M.A Lavrentiev during the WW II. This research was declassified in 1970-s and included in the 4-th edition of Lavrentiev–Shabat textbook of complex analysis.

As I don't know of any other place where this or similar theory is published, I decided to write this account in English. The projectiles based on the cumulative effect have no established name in English. They are known under various names like HEAT (high explosive anti tank), APHE (armor-piercing high explosive), hollow charges or shaped charges.

Mathematically this problem is modeled by collision of two jets of ideal liquids. It might sound a bit strange that both the jet and the *armor* are modeled as ideal liquids, but actually this is a good model. The collision speeds here are from 2 to 10 km/sec and the pressures are of the order of million atmospheres! In such conditions all elastic properties, as well as viscosity, become completely irrelevant.

Now we give a precise mathematical formulation of the problem. We have two colliding jets moving one towards another along the x-axis. Everything is symmetric with respect to rotation about the x-axis. The first jet has density ρ_1; it moves left to right from $-\infty$, has cross section of diameter $2r_1$ and speed V_1 at $-\infty$. The second jet of density ρ_2 moves right to left, from $+\infty$, has diameter $2r_2$ at $+\infty$.

The outer boundary of the first jet is formed by a curve L_1 (rotated around the x-axis, and the outer boundary of the second jet is formed by the curve L_2. In addition, we have a common boundary (where the jets collide) described by a curve γ (rotated about the x-axis). We assume that $0 \in \gamma$ and the speeds of both jets are zero at 0 (0 is the collision point on the x-axis).

There is some constant pressure outside the jets and some pressure p, depending on the point inside the jets. Bernoulli's law relates the pressure and the speed:

$$p = A - \frac{\rho}{2} V^2,$$

where A is a constant which is equal to the pressure at $x = 0$ on the x-axis, because $V = 0$ at this point by assumption. On the outer boundary we have

$$V = \text{const} = V_1 \quad \text{on} \quad L_1 \quad \text{and} \quad V = V_2 \quad \text{on} \quad L_2,$$

because the outside pressure is constant. It follows from Bernoulli's law that

$$V_2 = \sqrt{\frac{\rho_1}{\rho_2}} V_1. \tag{1}$$

On the separation surface, the pressures from both sides must be equal, so we obtain

$$\rho_1 V_+^2 = \rho_2 V_-^2, \tag{2}$$

where V_+ and V_- are the speeds on the left and right sides of γ.

All these considerations hold in both two- and three- dimensional models. Now we consider the problem in dimension 2.

Let $w = f_1(z) = u_1(z) + iv_1(z)$ and $f_2(z) = u_2(z) + iv_2(z)$ be the complex potentials of out two jets. Because of the symmetry it is sufficient to consider the upper halfs of the regions occupied by the jets. Each f_j maps conformally the upper half of the region occupied by the jet onto a horizontal strip. We choose these strips to be

$$0 < v < q_1 \quad \text{and} - q_2 < v < 0,$$

where $q_i = V_i r_i$ are intensities of the jets (volume passing through a cross section per second). We also assume that $f_i(0) = 0$.

The boundary correspondence is as follows: L_1 is mapped onto $v = q_1$, L_2 onto $v = -q_2$, the real axis onto two sides of the negative axis and the curve γ onto the positive axis.

Now we have

$$|f_1'(z)| = V_1, \quad z \in L_1, \tag{3}$$

and using (1):

$$|f_2'(z)| = V_2 = \sqrt{\frac{\rho_1}{\rho_2}} V_1, \quad z \in L_2, \tag{4}$$

and from (2) follows

$$|f_2'(z)| = \sqrt{\frac{\rho_1}{\rho_2}} |f_1'(z)|. \tag{5}$$

The positive and negative rays of the real line are mapped on the upper and lower sides of the negative ray by f_1 and f_2, respectively, so

$$\arg f_1'(z) = 0, \quad x < 0, \quad \text{and} \quad \arg f_2'(z) = -\pi, \quad x > 0. \tag{6}$$

Now we use the same method as in the simpler problem of collision of one jet with a fixed plane (also posted on this web site). To simplify the problem, let us consider the inverse functions $z_i(w) = f_i^{-1}(w)$, and set

$$\zeta = \log f_i'(z_i(w)) = F_i(w).$$

Then (5) and (6) give

$$\Re F_1(w) = \log V_1, \quad \Im w = q_1, \tag{7}$$

$$\Re F_2(w) = \log V_1 + (1/2) \log \rho_1/\rho_2. \tag{8}$$

On the positive ray we have

$$\Re F_2(w_2) = \Re F_2(w_1) + (1/2) \log \rho_1/\rho_2, \quad \Im F_2(w_2) = \Im F_1(w_1), \tag{9}$$

where $w_i = f_i(z)$, and on the negative ray

$$\Im F_1(x + i0) = 0, \quad \Im F_2(x - i0) = -\pi. \tag{10}$$

From (5) we conclude that $w_2 = \sqrt{\rho_1/\rho_2} w_1$, and then from (9) follows that the function

$$F_2\left(\frac{\rho_1}{\rho_2} w\right) - \frac{1}{2} \log \frac{\rho_1}{\rho_2}$$

is an analytic continuation of F_1 across the positive semi-axis.

Thus the problem is reduced to finding a function $F(w) = F_1(w)$ analytic in the strip

$$-q_2 \sqrt{\frac{\rho_1}{\rho_2}} < v < q_1$$

cut along the negative ray, which on the boundary of the strip satisfies

$$\Re F = \log V_1$$

and on the upper and lower edges of the cut respectively the conditions

$$\Im F(u + i0) = 0, \quad \text{and} \quad \Im F(u - i0) = -\pi, \quad u < 0.$$

Without loss of generality we may assume that $V_1 = 1$, then $q_1 = r_1$; and in view of (3) $q_2\sqrt{\rho_1/\rho_2} = r_2$. As from the physical meaning of the problem $\Re F(w)$ must be bounded from above, the function $\zeta = F(w)$ must perform the conformal map of the strip $-r_2 < v < r_1$ with the cut along the negative ray onto a half-strip. This function can be found using the Schwarz-Christoffel formula.

References

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[3] M. A. Lavrentiev, B. V. Shabat, Métodos de la teoria de las funciones de una variable compleja, Moscow, 1991.