Calculations and Assumptions for Penetration Model

Relatively simple one-dimensional expressions of hydrodynamics can be used to model the penetration and crater development (permanent cavity) of a deforming bullet in tissue. In the discussion that follows, certain assumptions are maintained:

- The projectile (bullet) deforms primarily by ductile erosion, rather than by fragmentation
- No significant yaw or pitch of the bullet is observed during penetration
- Energy not accounted for by the material failure criterion is adequately captured by the inertial and viscous drag losses (i.e., direct heating is insignificant and elastic responses are secondary, late time effects)

There are two distinct phases of behavior for a deforming bullet during a penetration event. In the first, called the hydrodynamic penetration phase, the bullet is subject to plastic deformation under the pressure felt at its interface with the tissue in the target. In the latter phase, called the rigid-body penetration phase, the bullet has ceased to deform plastically and penetrates without further change in shape or loss of mass. Different mechanical expressions are used to describe these two phases. Integration is performed in both phases in terms of velocity, a convenient quantity.

The method described does not explicitly account for the significant energy expended by bullet deformation and mass loss. However, the effect of these behaviors is otherwise described by adapting the Tate equations for a homogenous eroding rod in hydrodynamic penetration and by using a separate set of functions to calculate variables such as expanded diameter and retained mass as a function of the deformed length of the bullet. Since the Tate equations are well supported by analytical and empirical evidence, we may use them with confidence, and since the rigid body mechanics do depend explicitly on the kinetic energy (using retained mass and the known velocity), the results should be very good.

The principal expressions of the penetration mechanics in this method are analytical, which is a great strength of the method. Empirical values are introduced in the implementation to account for bullet expansion and mass loss, based (properly) on test results. The method accounts for inertial and viscous drag forces, material failure conditions, and strength effects at low velocities. While both penetration and cavity diameter are calculated, the model is truly only one-dimensional in the axial and radial dimensions (call it 1-and-a-half-dimensional); no meshing scheme is involved, nor are material strains calculated to determine cavity growth.

Tate Equations for Eroding Rods (Hydrodynamic Penetration)

The Tate equations were developed to describe the penetration of a solid, homogeneous cylindrical rod penetrating a homogeneous target. These equations are derived by using a momentum balance at the stagnation point and assuming that the non-deformed portion of the eroding rod behaves as a rigid body. Integration is performed in terms of decaying velocity. The Tate equations are adapted for use with core and jacket bullets by using a composite density and yield strength based on an area weighted function of these quantities for each cross-section of the bullet. Note that the penetration does not depend on the diameter or shape of the eroding projectile.

Differential penetration $dP = (\rho_p / \sigma_p) \, u \, dL \, dv$

where,

ρ_p is projectile density
σ_p is projectile yield strength
u is penetration velocity
dL is differential deformed projectile length
dv is differential projectile velocity

Penetration velocity $u = [1 / (1 - \gamma^2)] [v - \gamma (v^2 + A_0)^{1/2}]$

where,

$\gamma = (\rho_t / \rho_p)^{1/2}$
$A_0 = 2 (\sigma_t - \sigma_p) (1 - \gamma^2) / \rho_t$
ρ_t is target density
σ_t is target yield strength
v = instantaneous velocity

Differential deformed length

$$dL = L [A_1 / A_2]^{A_3} e^{[A_4 (A_5 - A_6)]}$$

where,

$A_1 = v + (v^2 + A_0)^{1/2}$
$A_2 = v_0 + (v_0^2 + A_0)^{1/2}$
$A_3 = (\sigma_t - \sigma_p) / (\gamma \sigma_p)$
$A_4 = \gamma \rho_p / [2 \sigma_p (1 - \gamma^2)]$
$A_5 = v + (v^2 + A_0)^{1/2} - \gamma v^2$
$A_6 = v_0 + (v_0^2 + A_0)^{1/2} - \gamma v_0^2$
v_0 = impact velocity

Hydrodynamic Termination Velocity

Hydrodynamic flow conditions cease at this velocity threshold, when the penetration velocity equals the velocity of the rear portion of the projectile. Note that the following relation is only valid where the strength of the projectile is greater than that of the target.

$v_{HT} = [2 (\sigma_p - \sigma_t) / \rho_T]^{1/2}$ where,

σ_p is projectile yield strength
σ_t is target yield strength
ρ_t is target density

Cranz' Law Assumption

The temporary cavitation is proportional to the kinetic energy expended in each differential increment of penetration, less losses to projectile deformation. In *non-hydrodynamic* penetration, both the temporary and the permanent cavitations are proportional to the kinetic energy expended (ie, no work is being done on the projectile).

Non-Hydrodynamic Penetration

After the termination of hydrodynamic penetration, the projectile continues to penetrate as a rigid body. The rigid body penetration is proportional to the kinetic energy and inversely proportional to the resistive forces acting on the deformed bullet. There are two principal sources of resistive forces: a quasi-static material failure force and a dynamic drag force.

Tissue is not a fluid; it is a solid and has strength. The quasi-static material failure force is that required to create a stress state in the target material sufficient to bring to yield or flow condition. This crushing force is proportional to the flow stress of the target medium and the diameter of the permanent crush cavity (which is a function of velocity and target material strength).

Flow stress criterion: $\sigma_t \sim E / V \rightarrow F_{CRUSH} = (\pi / 4)\, \sigma_t\, D_H^2$

where, D_H Diameter of the crushed volume (hole)

The dynamic drag force is that required to overcome the inertial and viscous drag in the failed target material and pass through. Because the Reynolds number is used to calculate the drag coefficient, both inertial and viscous drag forces are represented in the drag coefficient. In practice, the inertial drag forces dominate the problem. Even for small bullets at very low (termination) velocities, the Reynolds number will be greater than 100, so the boundary layer is small and laminar flow is not the dominant behavior. The model assumes that the drag over a sphere can be used. For the range of Reynolds numbers involved, this is a reasonable assumption. Alternatively, the drag on a hollow hemisphere (open downstream) might be better still.

Drag Force: $F_{DRAG} = (\pi / 4)\, D_X^2\, \rho_t\, (v^2 / 2)\, C_D$

where,	D_X	Expanded diameter
	$C_D = f(Re)$	Drag coefficient
	$Re = (\pi / 4)\, (\rho_t / \mu)\, v\, D_X$	Reynolds number
	μ	Dynamic viscosity

The combined forces acting on the projectile can be resolved into the following expression for penetration (integrated in terms of velocity):

Rigid-Body Penetration:

$$dP = dE / \{ (\pi / 4)\, \sigma_t\, D_H^2\, [1 + (\rho_t\, D_X^2\, C_D\, v^2 / (4\, D_H^2 \sigma_t))] \}$$

where,	$dE = m_R\, (v_1^2 - v_2^2) / 2$	Differential energy
	m_R	Retained projectile mass

Elastic Limit Velocity

Determines the lower limit of rigid body penetration based on the dynamic pressure of the projectile (assumes a blunt shape). Elastic limit is the velocity at which the dynamic pressure equals the yield stress of the target material.

$$v_{EL} = [\ 2\ \sigma_t\ /\ \rho_p\ ]^{1/2}$$

Permanent Cavity Diameter

From the Held equation for permanent crater dimensions in jet penetration:

$$D_H = D_X\ v\ \{\ \rho_p\ /\ [\ 2\ \sigma_t\ (\ 1 + 1\ /\ \gamma\ )^2\ ]\ \}^{1/2}$$

Model Limitations and Known Issues

The Tate eroding rod model is quite sensitive to the initial length of the penetrator (since all the equations are expressed as a ratio relative to the initial length). This is somewhat problematic in that you must decide whether it makes sense to always measure the total projectile length (the yield strength, retained mass and expanded diameter functions are all derived from a deformed length function) or to estimate the total possible deformable length (which for monolithics, such as the Barnes X-Bullet, is a fixed value less than half the total length). Whatever approach is applied, it is imperative that the assumptions be clearly recognized in developing the functions.

The method described does not account for the partitioning of the kinetic energy between elastic and plastic strain, however, the Held equation is probably very accurate for all but the velocities near the termination point (elastic limit velocity).

Appendix: Alternative Implementation for Universal Bullet Expansion

The following method for calculating the expanded diameter (D_X) as a function of the deformed length (dL) may be used with reasonable success if specific data for bullet design behavior is not known. The premise is that the deformed bullet can be treated incrementally as an equivalent cylinder that is flattened into a coin of metal. Then this flat area is assumed to be equivalent to the area of a hemispheric bullet mushroom and finally a presented (2-dimensional circular) area is derived from the spherical area.

Cylindrical deformation assumption

Each differential cylindrical segment that is deformed can be described as a ring within a total flattened coin of discrete nominal thickness. For any total deformed length, the area of the deformed coin is given by the original cylindrical volume and the nominal deformed thickness (assumed).

Hemispherical assumption

The presented (circular) area of the mushroom deformed projectile is ½ that of the spherical area (i.e., the presented area is a circular 2-dimensional projection of the 3-dimensional surface of a sphere). The spherical area is approximated by the area of the deformed cylindrical length (see hereafter).

Undeformed cylinder	D, L	Diameter, length
Deformed coin	A, h	Area, thickness
Equivalence assumption	$A = A_S$	Cylindrical area same as spherical
Spherical area	$A_S = 2 \pi r^2$	in terms of radius
	$A_S = \pi/2\, D_X^2$	in terms of expanded diameter
Volume = constant	$\pi/4\, D^2 (L - dL) = A h$	Cylindrical
	$\pi/4\, D^2 (L - dL) = A_S h$	Spherical
	$\pi/4\, D^2 (L - dL) = \pi/2\, D_X^2 h$	
Expanded diameter	$D_X = [(D^2 (L - dL)) / (2 h)]^{1/2}$	